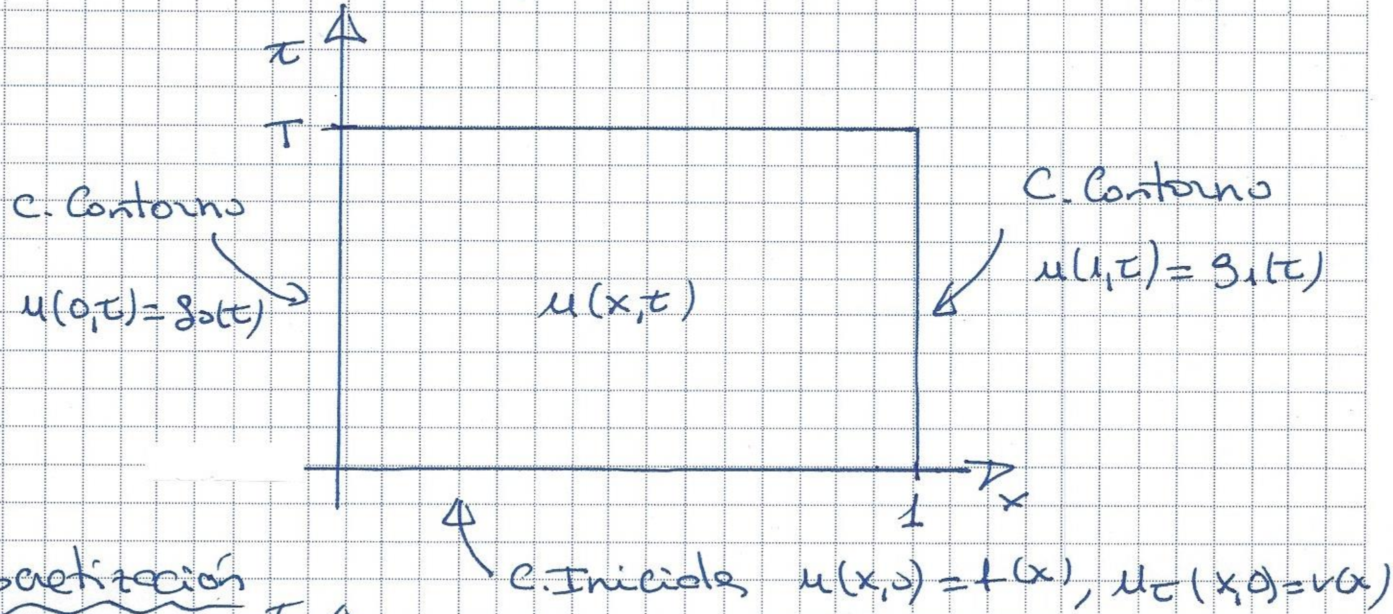
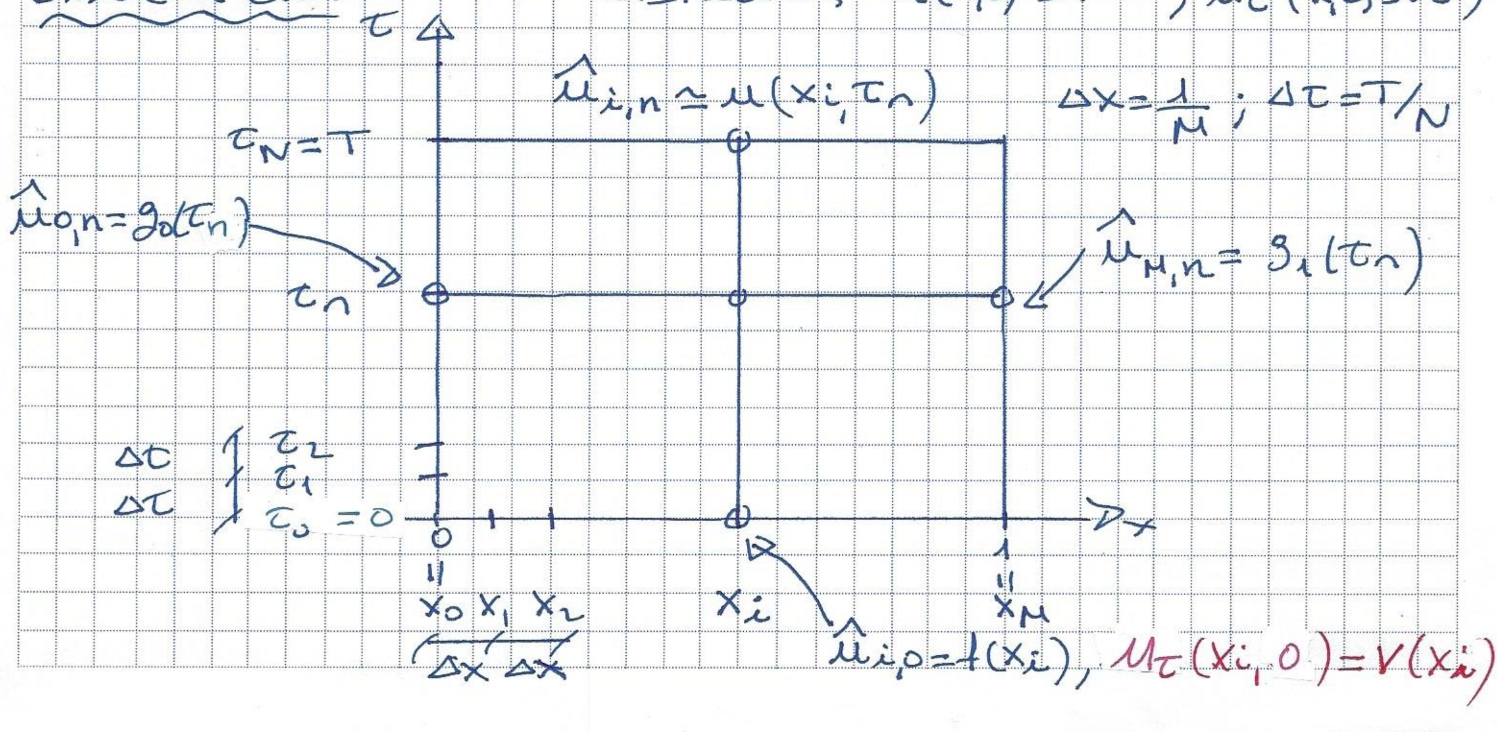
ECUACIONES HIPERBÓLICAS 1D

$$
\begin{cases}
u_{tt} = u_{xx} & ; 0 < x < 1 \\
 & ; 0 < t < T \quad \text{(EDP)} \\
\begin{cases} u(x,0) = f(x) & ; 0 \leq x \leq 1 \quad \text{(C. Inicial)} \\ u_t(x,0) = v(x) & ; 0 \leq x \leq 1 \quad \text{(C. Inicial)} \end{cases} \\
\begin{cases} u(0,t) = g_0(t) & ; 0 < t \leq T \quad \text{(C. Contorno)} \\ u(1,t) = g_1(t) & ; 0 < t \leq T \quad \text{(C. Contorno)} \end{cases}
\end{cases}
$$



Discretización



1) Integración numérica de la EDP

$$\left[u_{\tau\tau} - u_{xx}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = 0$$

$$\begin{cases} u_{\tau\tau}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_i,\tau_{n+1}) - 2u(x_i,\tau_n) + u(x_i,\tau_{n-1})}{\Delta\tau^2} - u_{\tau\tau\tau\tau}\dfrac{\Delta\tau^2}{12} + O(\Delta\tau^4) \\[2ex] u_{xx}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_{i+1},\tau_n) - 2u(x_i,\tau_n) + u(x_{i-1},\tau_n)}{\Delta x^2} - u_{xxxx}\dfrac{\Delta x^2}{12} + O(\Delta x^4) \end{cases}$$

$$\boxed{\begin{gathered} \frac{u(x_i,\tau_{n+1}) - 2u(x_i,\tau_n) + u(x_i,\tau_{n-1})}{\Delta\tau^2} - \frac{u(x_{i+1},\tau_n) - 2u(x_i,\tau_n) + u(x_{i-1},\tau_n)}{\Delta x^2} = \tau(x_i,\tau_n) \\[2ex] \tau(x_i,\tau_n) = \left[u_{\tau\tau\tau\tau}\frac{\Delta\tau^2}{12} - u_{xxxx}\frac{\Delta x^2}{12}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} + O(\Delta\tau^4) + O(\Delta x^4) \end{gathered}}$$

$$\begin{cases} \text{Se define } \mu = (\Delta\tau/\Delta x)^2 \\[1ex] u_{\tau\tau} = u_{xx} \Rightarrow u_{\tau\tau\tau\tau} = u_{xx\tau\tau} = u_{\tau\tau xx} = u_{xxxx} \end{cases}$$

Luego: $\tau(x_i,\tau_n) = \dfrac{u_{xxxx}}{12}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} (\mu-1)\Delta x^2 + O(\Delta x^4\mu^2) + O(\Delta x^4)$

$$\boxed{\begin{gathered} \begin{cases} u(x_i,\tau_{n+1}) = -u(x_i,\tau_{n-1}) + \left(\mu u(x_{i+1},\tau_n) + 2(1-\mu)u(x_i,\tau_n) + \mu u(x_{i-1},\tau_n)\right) + \\ \qquad + \Delta\tau^2\tau(x_i,\tau_n) \\[1ex] \hat{u}_{i,n+1} = -\hat{u}_{i,n-1} + \left(\mu\hat{u}_{i+1,n} + 2(1-\mu)\hat{u}_{i,n} + \mu\hat{u}_{i-1,n}\right) \end{cases} \\[2ex] \hat{u}_{i,0} = f(x_i), \quad u_\tau(x_i,0) = v(x_i) \quad ; i = 0,\dots,M \\[1ex] \hat{u}_{0,n+1} = g_0(\tau_{n+1}), \quad \hat{u}_{M,n+1} = g_1(\tau_{n+1}) \quad ; n = 0,\dots,N-1 \end{gathered}}$$

Creamos una fila de nodos "fantasma" para $\tau_{-1} = -\Delta\tau$:

$$\left.\begin{array}{l} u_\tau(x_i,\tau_0) = \dfrac{u(x_i,\tau_1) - u(x_i,\tau_{-1})}{2\Delta\tau} + O(\Delta\tau^2) \\ \\ u_\tau(x_i,\tau_0) = v(x_i) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{u(x_i,\tau_1) - u(x_i,\tau_{-1})}{2\Delta\tau} - v(x_i) + O(\Delta\tau^2) = 0 \\ \dfrac{\hat{u}_{i,1} - \hat{u}_{i,-1}}{2\Delta\tau} - v(x_i) = 0 \end{array}}$$

Planteamos la EDP para $n=0$

$$\hat{u}_{i,1} = -\hat{u}_{i,-1} + \Big(\mu \underbrace{\hat{u}_{i+1,0}}_{f(x_{i+1})} + 2(1-\mu)\underbrace{\hat{u}_{i,0}}_{f(x_i)} + \mu \underbrace{\hat{u}_{i-1,0}}_{f(x_{i-1})}\Big)$$

Además: $\hat{u}_{i,1} = \hat{u}_{i,-1} + 2\Delta\tau\, v(x_i)$

Luego: $2\hat{u}_{i,1} = 2\Delta\tau\, v(x_i) + \mu f(x_{i+1}) + 2(1-\mu) f(x_i) + \mu f(x_{i-1})$

$$\Rightarrow \hat{u}_{i,1} = \Delta\tau\, v(x_i) + \frac{\mu}{2} f(x_{i+1}) + (1-\mu) f(x_i) + \frac{\mu}{2} f(x_{i-1})$$

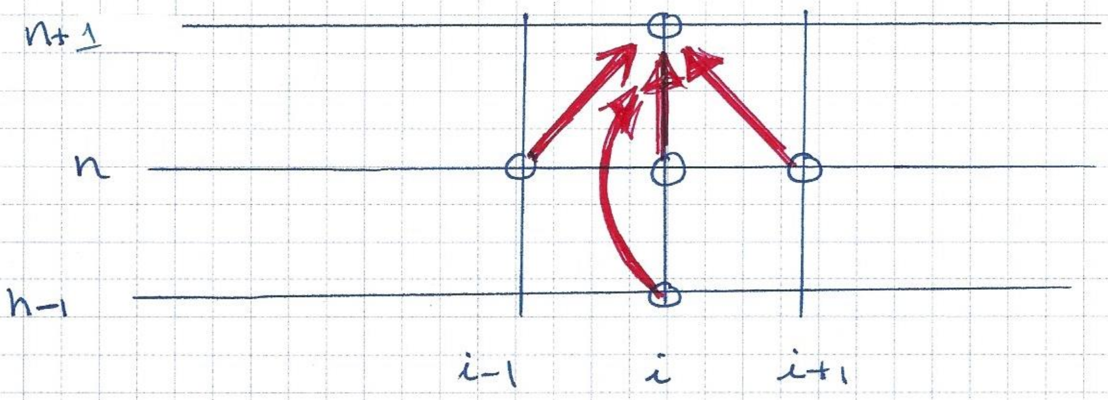
LEAPFROG (*)

$$n=0 \rightarrow \begin{cases} \hat{u}_{0,1} = g_0(t_1), \quad \hat{u}_{M,1} = g_1(t_1) \\ \hat{u}_{i,1} = \Delta t\, v(x_i) + \frac{\mu}{2} f(x_{i+1}) + (1-\mu) f(x_i) + \frac{\mu}{2} f(x_{i-1}) \end{cases}$$

$$i = 1, \ldots, M-1$$

$$n>0 \rightarrow \begin{cases} \hat{u}_{0,n+1} = g_0(t_{n+1}), \quad \hat{u}_{M,n+1} = g_1(t_{n+1}) \\ \hat{u}_{i,n+1} = -\hat{u}_{i,n-1} + \left(\mu \hat{u}_{i+1,n} + 2(1-\mu)\hat{u}_{i,n} + \mu\, \hat{u}_{i-1,n}\right) \end{cases}$$

$$i = 1, \ldots, M-1$$



(*) Pídola, o saltolarana en español

2) Análisis Modal

Buscamos soluciones tipo:

$$u(x,t) = \psi(x)\, e^{j\omega t} \Rightarrow \begin{cases} u_{xx} = \psi_{xx}\, e^{j\omega t} \\ u_{tt} = -\omega^2 \psi\, e^{j\omega t} \end{cases}$$

Luego $u_{tt} = u_{xx} \rightarrow -\omega^2 \psi e^{j\omega t} = \psi_{xx} e^{j\omega t}$

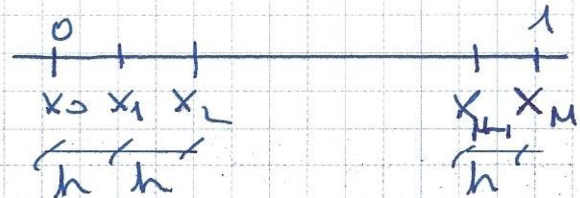
$$(\psi_{xx} + \omega^2 \psi)\, e^{j\omega t} = 0 \quad \forall t$$

$$\Rightarrow \boxed{\psi_{xx} + \omega^2 \psi = 0} \; ; \; 0 < x < 1$$

Problema de autovalores:

$$\boxed{\psi_{xx} = -\omega^2 \psi \quad ; \quad 0 < x < 1}$$

Resolvemos: $\begin{cases} \psi_{xx} = -\omega^2 \psi \\ \psi(0) = 0 \,,\; \psi(1) = 0 \end{cases}$



$$\boxed{\underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{M-1} \end{Bmatrix}} = \lambda \underbrace{\begin{Bmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{M-1} \end{Bmatrix}} \; ; \; \lambda = -\omega^2 h^2}$$

$\rightsquigarrow \underset{\sim}{A}\, \bar{\Psi} = \lambda\, \bar{\Psi}$

Resolviendo el problema de autovalores, obtendremos

$$\{(\lambda_i, \overline{\Psi}_i)\}_{i=1,\dots,M-1} \rightsquigarrow \omega_i = \sqrt{-\lambda_i/h^2}$$

$$\Rightarrow \begin{cases} \omega_i \equiv \text{frecuencias propias} \\ \overline{\Psi}_i \equiv \text{modos propios} \end{cases}$$

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