Basic aspects on drawing curves (09.02.2023)

These notes are intended to give some basic notions about drawing curves defined by Cartesian –explicit or parametric– and polar equations.

1 Curves in explicit Cartesian coordinates

The explicit Cartesian equation of a curve is of the form $y = f(x)$, where f is a function. To represent curves of this type it is useful to study the following aspects.

1.1 Domain

The first step usually consists in determining the domain of the function, identifying the points at which f is not defined. For example, if $f(x) = \sqrt{x^2 + x - 2}$, the function does not exist for values of $x \in (-2, 1)$.

1.2 Zeros and symmetries

a) We look for the values of x that make $f(x) = 0$, in which the curve intersects the OX axis.

b) If $f(-x) = f(x)$ (f is even), the curve is symmetric about the OY axis. E.g. $y = \cos x$.

c) If $f(-x) = -f(x)$ (f is odd), the curve is symmetric about the origin O. E.g. $y = x^3$.

1.3 Asymptotes

An asymptote is a line to which the curve gets as close as we want, without actually touching it (tangent at infinity). They can be of three types.

a) Vertical.

They appear if $f(x) \to \pm\infty$, when $x \to a$. Example: $y = (x^2 - 4)^{-1}$ has asymptotes at $x = \pm 2$, values which would make the denominator equal to zero.

The following curves also have vertical asymptotes: $y = \ln x$ at $x = 0^+$; and $y = \tan x$ at $x = (2k - 1)\pi/2,\ k \in \mathbb{Z}$.

b) Horizontal. There is a horizontal asymptote if $f(x) \to b$, when $x \to \pm\infty$. Example: in the curve $y = \dfrac{x}{x^2 + 1}$, the asymptote is the OX axis, since $y \to 0^\pm$ when $x \to \pm\infty$.

c) Oblique. There is an oblique (or slant) asymptote of equation $y = mx + n$ if, when $x \to \pm\infty$, then $y \to \pm\infty$ (or to $\mp\infty$) and we also have that

$$\lim_{x \to \pm\infty} \frac{y}{x} = m \in \mathbb{R}; \quad \lim_{x \to \pm\infty} y - mx = n \in \mathbb{R}$$

For example, the curve of equation $y = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$ has $y = 2x + 1$ as an asymptote.

1.4 Maxima, minima and inflection points

We study the values of the first and second derivatives of f, resulting in:

a) Maximum. If $f'(x_0) = 0,\ f''(x_0) < 0$, the function has a local maximum at $x = x_0$.

b) Minimum. If $f'(x_0) = 0,\ f''(x_0) > 0$, the function has a local minimum at $x = x_0$.

c) Inflection point. If $f''(x_0) = 0,\ f'''(x_0) \neq 0$, there is an inflection point at $x = x_0$.

1.5 Particular case of rational functions

The function $f(x)$ is given by a quotient of polynomials $y = \dfrac{P(x)}{Q(x)}$.

a) **Zeros.** The zeros of $f(x)$ are the roots of $P(x)$.

b) **Asymptotes.** The curve will have a vertical asymptote at the points corresponding to the roots of $Q(x)$. If the order of multiplicity of the root is even, the sign of $f(x)$ will not change on either side of the asymptote, and it will if the order is odd.

Examples: the curve $y = (x-1)^{-2}$ has a vertical asymptote at $x = 1$ with no sign change, while $y = (x-2)^{-3}$ has one, at $x = 2$, with a change of sign.

1.6 Proposed exercises

a) Study the curve of equation $y = x(x^2-1)^{-1}$. Check that it is symmetric about the origin of coordinates and has two vertical asymptotes and one horizontal.

b) Study the curve given by $y = (x^2+x-2)(x-2)^{-1}$. Check that it has two zeros, extrema at $x = 0$ and $x = 4$, one vertical asymptote and another oblique.

c) Study the curve given by $y = x + x^{-1}$. Check that it has extrema at $x = \pm 1$, one vertical asymptote and another oblique. Notice that the curve is symmetric about the origin O.

d) Study the curve of equation $y = \sqrt{\dfrac{x-1}{x}}$, noticing that it has two asymptotes (vertical and horizontal) and that it is not defined on a certain interval .

e) Plot the curve of equation $y = e^{-1/x}$, paying attention to the limits of $f(x)$ when $x \to 0^{\pm}$ and $x \to \pm\infty$.

f) Represent the following curves (solution in figures 1 to 4 of these notes):

$$y = \frac{x^4}{x-1}; \qquad y = \frac{x^2(x+1)^2}{x-1}; \qquad y = \sqrt[3]{\frac{x^4}{x-1}}; \qquad y = \sqrt[3]{\frac{x^2(x+1)^2}{x-1}}$$

2 Curves in parametric cartesian coordinates

In this type of curves, the coordinates (x, y) of the points of the curve are expressed as a function of a parameter t:

$$x = f(t); \quad y = g(t)$$

Giving values to t we obtain the different points. These curves do not always represent functions, since the same value of t can give rise to one value of x and several values of y.

To represent these curves we will analyze the aspects mentioned in section **??**, for which we must determine certain values of the parameter.

2.1 Intersection with the coordinate axes

a) Intersection with OX. We look for the values of t that make $g(t) = 0$:

$$t = t_1 \,/\, g(t_1) = 0 \Longrightarrow \text{point } P_1\,(f(t_1), 0)$$

b) Intersection with OY. We look for the values of t that make $f(t) = 0$:

$$t = t_2 \,/\, f(t_2) = 0 \Longrightarrow \text{point } P_2\,(0, g(t_2))$$

2.2 Asymptotes

a) Vertical asymptote at $x = a$. We look for the values of t that satisfy

$$t = t_3 \,/\, f(t_3) = a, \quad \lim_{t \to t_3} g(t) = \pm\infty$$

b) Horizontal asymptote of ordinate $y = b$. We look for the values of t that satisfy

$$t = t_4 \,/\, \lim_{t \to t_4} f(t) = \pm\infty, \quad g(t_4) = b$$

c) Oblique asymptote of equation $y = mx + n$. We look for the values of t that satisfy

$$t = t_5 \,/\, \lim_{t \to t_5} f(t) = \pm\infty, \quad \lim_{t \to t_5} g(t) = \pm\infty$$

as well as

$$\lim_{t \to t_5} \frac{f(t)}{g(t)} = m \in \mathbb{R}; \quad \lim_{t \to t_5} g(t) - mf(t) = n \in \mathbb{R}$$

2.3 Tangents

From the equations $x = f(t)$, $y = g(t)$ it results $dx = f'(t)$, $dy = g'(t)dt$, which allows us to obtain the condition for the points of horizontal tangent

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = 0$$

or vertical tangent

$$\frac{dx}{dy} = \frac{f'(t)}{g'(t)} = 0$$

2.4 Double points

Double points are those through which the curve passes twice. A double point is obtained when, for two distinct values of t, both the corresponding values of x and y coincide.

$$\exists\, t_6, t_7 \,/\, f(t_6) = f(t_7), \quad g(t_6) = g(t_7)$$

2.5 Representation of the curves $x = f(t)$ and $y = g(t)$

To facilitate the location of the values of t mentioned in the previous sections, it may be useful to previously draw the curves of equation $x = f(t)$ (on axes $t - x$) and $y = g(t)$ (on axes $t - y$).

2.6 Examples

a) In figures 5 and 6 the following curves are represented.

$$x = \frac{t(t-1)(t-2)}{t+1}, \; y = \frac{1}{t-1}; \qquad x = \frac{(t^2)(t-1)}{t+1}, \; y = \frac{t^2}{t+1}$$

For each of them, the $t - x$ curve, the $t - y$ curve and the $x - y$ curve are shown.

b) Figure 7 represents the cycloid, of equation $x = t - \sin t$, $y = 1 - \cos t$. In this case, t takes values in the interval $[0, 2\pi]$, thus giving rise to a single arc.

c) Figure 8 represents the astroid, of equation $x = 2\cos^3 t$, $y = 2\sin^3 t$. Obtain its equation in coordinates $x - y$, eliminating the parameter t.

d) Figure 9 represents the circle of parametric equations $x = 3\cos^2 t$, $y = 3\sin t \cos t$. Obtain its equation in coordinates $x - y$, which allows determining the radius and the position of the center without having to draw it.

3 Curves in polar coordinates

3.1 Definition

Given a point $P(x,y)$ in the plane, the oriented segment that joins the origin with P is called the radius vector and its length is denoted by ρ. The angle formed by the radius vector with the positive direction of the OX axis is denoted by θ, taking the counterclockwise direction of rotation as positive.

The polar coordinates of a point are (ρ,θ). The origin of coordinates is called the pole. The OX axis is the polar axis.

3.2 Relation between polar and cartesian coordinates

Projecting the radius vector on the axes OX and OY, we see that the coordinates x and y of P correspond to the values $\rho\cos\theta$ and $\rho\sin\theta$ respectively. To obtain the inverse relation between both coordinate systems, we do the following

$$x^2+y^2=\rho^2(\cos^2\theta+\sin^2\theta)=\rho^2,\ \ \tan\theta=\frac{y}{x}\ (x\neq 0)$$

from where

$$\rho=\sqrt{x^2+y^2}\ ,\quad \theta=\operatorname{arctg}\frac{y}{x}\ (x\neq 0)$$

The angle θ can take any real value. Those in the interval $(-\pi,\pi]$ are called principal values. In polar coordinates, a curve is defined by a relation $\rho=\rho(\theta)$.

3.3 Examples

a) The equation of the circle of center $C(R,0)$ and radius R, is $\rho=2R\cos\theta$. Figure 9 (curves in parametric coordinates) represents the case $R=1.5$.

b) Equations of the form $\rho=a\cos n\theta,\ a\in\mathbb{R}^+,\ n\in\mathbb{N}$, are called "$n-$petal roses". The value of a determines the size of the petals (figs. 10 and 11).

c) The cardioid has the equation $\rho=a\,(1+\cos\theta)$ (in figure 12, $a=3$).

d) The equation of the Archimedean spiral is $\rho=a\theta$, so the length of the radius vector is null for $\theta=0$ and the curve passes through the pole. We notice that, in each turn (θ grows by 2π), ρ increases by the amount $a\,2\pi$. In the example shown in fig. 13, $a=3$.

e) Figure 14 represents the Lemniscate of Bernoulli $\rho^2=\cos 2\theta$.

f) In the case of the asymptotic circle (fig. 15) it can be seen that, for values of $\theta\to 0$, the length of the radius vector $\rho\to\infty$ and an horizontal asymptote appears. And when $\theta\to\infty$, the length $\rho\to 1$ and the points (ρ,θ) approach the circumference of radius 1, which explains the name of the curve.