LINEAR ALGEBRA II — Solutions to practice 4.2

Quadric — (Course 2022–2023)

NOTE: This is a mostly unrevised, automatic translation from Spanish.

NOTE: All the problems are supposed to be posed in the Eucídean affine space endowed with a rectangular Cartesian system.

1.– *Given the quadric equation:*

$$2x^2 - y^2 - z^2 - 2xy + 2xz - 4yz - 2x + 2y = 0.$$

classify the surface and sketch a picture of it. The matrix associated with the quadric is:

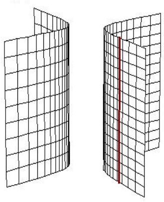
$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

To classify it, we diagonalize it by congruence taking into account that the last row cannot be changed position, multiplied by a number or added to the others:

$$A \overset{H_{13}}{\longrightarrow} \overset{\mu_{13}}{\longrightarrow} \begin{pmatrix} -1 & -2 & 1 & 0 \\ -2 & -1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \overset{H_{21}(-2)}{\longrightarrow} \overset{H_{31}(2)}{\longrightarrow} \overset{\mu_{21}(-2)}{\longrightarrow} \overset{\mu_{31}(2)}{\longrightarrow} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \longrightarrow$$

$$\overset{H_{32}(1)}{\longrightarrow} \overset{H_{42}(-1/3)}{\longrightarrow} \overset{\mu_{32}(1)}{\longrightarrow} \overset{\mu_{42}(-1/3)}{\longrightarrow} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/3 \end{pmatrix}.$$

We see that the matrix diagonalzia; has rank 3 and its signature is $(-, +, 0; -)$ or equivalently $(+, -, 0; -)$. It is therefore a hyperbolic cylinder.



2.– *Given the equation:*

$$x^2 + 2y^2 + 5z^2 + 2xy + 2xz + 2yz + 4x + 2y + 4z + 1 = 0,$$

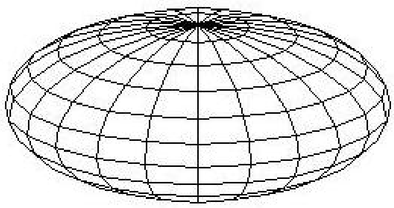
classify the quadric it defines and sketch a picture of it. The matrix associated with the quadric is:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

To classify it, we diagonalize it by congruence, taking into account that for the elementary operations to represent a change of reference affín the last row cannot be moved, added to others or multiplied by a number.

$$A \overset{\mu_{21}(-1)}{\underset{H_{21}(-1)}{\longrightarrow}} \overset{\mu_{31}(-1)}{\underset{H_{31}(-1)}{\longrightarrow}} \overset{\mu_{41}(-2)}{\underset{H_{41}(-2)}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & -3 \end{pmatrix} \overset{\mu_{42}(1)}{\underset{H_{41}(1)}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

The distribution of signs in diagonal form is $(+,+,+;-)$. It is therefore a real ellipsoid.



3.– Write the equation of:

(a) *a non-degenerate quadric that does not contain ellipses.* A hyperbolic paraboloid:

$$x^2 - y^2 - 2z = 0.$$

(b) *a non-degenerate quadric containing ellipses and infinitely many lines.* A one-sheet hyperboloid:

$$x^2 + y^2 - z^2 - 1 = 0.$$

(c) *a quadric containing ellipses, parabolas, and hyperbolas.* A real cone:

$$x^2 + y^2 - z^2 = 0.$$

4.– *Given the quadric equation:*

$$3x^2 + 2y^2 - z^2 + 4xy + 2xz + 4x - 4z - 1 = 0.$$

classify the surface and sketch a picture of it. The matrix associated with the quadric is:

$$A = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & -1 & -2 \\ 2 & 0 & -2 & -1 \end{pmatrix}.$$

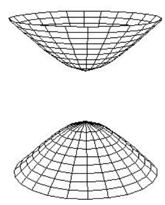
To classify it, we diagonalize it by congruence, taking into account that the ĺast row cannot be added to the others, nor can its position be changed, nor can it be multiplied by a number.

$$A \xrightarrow{H_{13}}\xrightarrow{\mu_{13}} \begin{pmatrix} -1 & 0 & 1 & -2 \\ 0 & 2 & 2 & 0 \\ 1 & 2 & 3 & -2 \\ -2 & 0 & 2 & -1 \end{pmatrix} \xrightarrow{H_{31}(1)}\xrightarrow{H_{41}(-2)}\xrightarrow{\mu_{31}(1)}\xrightarrow{\mu_{41}(-2)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{H_{32}(-1)}\xrightarrow{\mu_{32}(-1)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The matrix diagonalizes and the signature obtained is:

$$(-,+,+;+) \iff (+,+,-;+)$$

It is therefore a hyperboloid with two sheets.



5.– *Given the quadric of equation:*

$$x^2 + 2y^2 + z^2 + 6xz - 2x + 4y - 6z + 3 = 0.$$

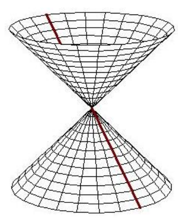
i) *Classify the surface and sketch a picture of it.* The matrix associated with the quadric is:

$$A = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 1 & -3 \\ -1 & 2 & -3 & 3 \end{pmatrix}$$

To classify the quadric we diagonalize it by congruence:

$$A \xrightarrow{H_{31}(-3)}\xrightarrow{H_{41}(1)}\xrightarrow{\mu_{31}(-3)}\xrightarrow{\mu_{41}(1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -8 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \xrightarrow{H_{42}(-1)}\xrightarrow{\mu_{42}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The signature is $(+,+,-;0)$. It is therefore a real cone.



ii) *¿Is there a plane that cuts the surface in a parabola?.* YES. We know that the conics are precisely the curves that are obtained by cutting a cone with a plane. Therefore any non-degenerate conic can be obtained as a cut of a plane with a cone. In particular, parabolas appear if we take a plane parallel to a generatrix and that does not pass through the vertex.

6.– *We consider the family of quadrics of* $\mathbb{R}^3$*:*

$$Q_{\alpha,\beta} : x^2 + \alpha z^2 + 2\beta x + 2\beta y + 2\beta z = 0$$

Classify according to α *and* β *the different quadrics that can appear.* The matrices associated with these are:

$$A_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \alpha & \beta \\ \beta & \beta & \beta & 0 \end{pmatrix}.$$

To classify them, we diagonalize them by congruence, taking into account that the ĺast row cannot be changed in position, nor added to the others, nor multiplied by a number:

$$A_{\alpha,\beta} \underset{\longrightarrow}{\overset{H_{41}(-\beta)}{\nu_{41}(-\beta)}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \alpha & \beta \\ 0 & \beta & \beta & -\beta^2 \end{pmatrix}.$$

If $\beta \neq 0$ cannot be diagonalized. The type of quadric depends on the signature of the matrix of quadratic terms:

$$T_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Where: - If $\alpha > 0$, the signature of T is $(+, +, 0)$ and it is an elliptic paraboloidí. - If $\alpha = 0$, the signature of T is $(+, 0, 0)$ and it is a parabolic cylinder. - If $\alpha < 0$, the signature of T is $(+, -, 0)$ and it is a hyperbolic paraboloid. If $\beta = 0$, the matrix $A_{\alpha,0}$ becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now: - If $\alpha > 0$ it is two imaginary planes intersecting in a real line. - If $\alpha = 0$ it is a double plane. - If $\alpha < 0$ it is about two real planes intersecting in a straight line. We summarize the classification in the following table:

	$\beta = 0$	$\beta \neq 0$
$\alpha > 0$	elliptic paraboloid	Imaginary intersecting planes
$\alpha = 0$	Parabolic cylinder	Double plane
$\alpha < 0$	Hyperbolic paraboloid	Real intersecting planes

(Final exam, December 2005)

7.– *In Eucídean space and with respect to a rectangular reference, they are considered the quadrics that admit by equations:*

$$x^2 - 2y^2 + az^2 - 2xz + 2yz + 2x + 1 = 0, \qquad \textit{with} \qquad a \in \mathbb{R}.$$

Classify said squares according to the different values of a. We write the associated matrix and diagonalize it for congruence, taking into account that the ĺast row cannot be moved or added to the othersás:

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & a & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \underset{\longrightarrow}{\overset{H_{31}(1)}{H_{41}(-1)}} \underset{\longrightarrow}{\overset{\mu_{31}(1)}{\mu_{41}(-1)}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underset{\longrightarrow}{H_{32}(1/2)} \underset{\longrightarrow}{\mu_{32}(1/2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & a-1/2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If $a \neq 1/2$

can be further diagonalized:

$$\overset{H_{32}(-1/(a-1/2))}{\longrightarrow}\overset{\mu_{32}(-1/(a-1/2))}{\longrightarrow}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & a-1/2 & 0 \\ 0 & 0 & 0 & -1/(a-1/2) \end{pmatrix}$$

We see as varían the signs of the diagonal terms: - If $a < 1/2$, then the signs are $(+,-,-;+)$ or equivalently $(+,+,-;-)$. It is a one sheet hyperboloid. - If $a > 1/2$, then the signs are $(+,+,-;-)$. It is a one sheet hyperboloid. - If $a = 1/2$, it is not possible to continue diagonalizing. The signature of the matrix of quadratic terms is $(+,-,0)$. It is a hyperbolic paraboloid. **(Final exam, September 2006)**

8.– *In the space affíy with respect to a rectangular reference, the quadrics of equations are considered:*

$$ax^2 + (1-a)y^2 + az^2 + 2(1-a)xz + 2x + 2z + 3 = 0,$$

with $a \in \mathbb{R}$. Classify quadrics based on parameter a. The matrix associated to the quadric is:

$$A = \begin{pmatrix} a & 0 & 1-a & 1 \\ 0 & 1-a & 0 & 0 \\ 1-a & 0 & a & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

To classify it we make congruence taking into account that the last row cannot be moved or added to the others.

$$A \overset{H_{12}}{\longrightarrow}\overset{\nu_{12}}{\longrightarrow} \begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & a & 1-a & 1 \\ 0 & 1-a & a & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix} \overset{H_{32}(1)}{\longrightarrow}\overset{\nu_{32}(1)}{\longrightarrow} \begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & a & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \overset{H_{23}}{\longrightarrow}\overset{\nu_{23}}{\longrightarrow}$$

$$\longrightarrow \begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & to & 1 \\ 0 & 2 & 1 & 3 \end{pmatrix} \overset{H_{32}(-1/2)H_{42}(-1)}{\longrightarrow}\overset{\mu_{32}(-1/2)\mu_{42}(-1)}{\longrightarrow} \begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & a-1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We distinguish the different cases depending on the values of a for which the terms of the diagonal vanish: - If $a < 1/2$ then the signature is $(+,+,-;+)$. It is a hyperboloid with two sheets. - If $a = 1/2$ then the signature is $(+,+,0;+)$. It is an imaginary cylinder. - If $1/2 < a < 1$ then the signature is $(+,+,+;+)$. It is an imaginary ellipse. - If $a = 1$ then the signature is $(+,+,0;+)$. It is an imaginary cylinder. - If $a > 1$ then the signature is $(+,+,-;+)$. It is a hyperboloid with two sheets.

9.– *Classify, according to the parameter λ, the quadric:*

$$(4-\lambda)x^2 + 2y^2 - \lambda z^2 + 4xy + 2\lambda xz + 4x - 4z - \lambda = 0.$$

The associated matrix is:

$$A = \begin{pmatrix} 4-\lambda & 2 & \lambda & 2 \\ 2 & 2 & 0 & 0 \\ \lambda & 0 & -\lambda & -2 \\ 2 & 0 & -2 & -\lambda \end{pmatrix}$$

We try to diagonalize it for congruence taking into account that the ĺast row cannot be changed position, nor multiplied by a number, nor added to the others:

$$A \overset{H_{12}}{\longrightarrow}\overset{\nu_{12}}{\longrightarrow} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 4-\lambda & \lambda & 2 \\ 0 & \lambda & -\lambda & -2 \\ 0 & 2 & -2 & -\lambda \end{pmatrix} \overset{H_{21}(-1)}{\longrightarrow}\overset{\nu_{21}(-1)}{\longrightarrow} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2-\lambda & \lambda & 2 \\ 0 & \lambda & -\lambda & -2 \\ 0 & 2 & -2 & -\lambda \end{pmatrix} \overset{H_{23}}{\longrightarrow}\overset{\nu_{23}}{\longrightarrow}$$

$$\longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & -2 \\ 0 & \lambda & 2-\lambda & 2 \\ 0 & -2 & 2 & -\lambda \end{pmatrix} \xrightarrow{H_{32}(1)} \xrightarrow{\nu_{32}(1)} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & -\lambda \end{pmatrix} \xrightarrow{H_{23}} \xrightarrow{\nu_{23}} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -\lambda & -2 \\ 0 & 0 & -2 & -\lambda \end{pmatrix}$$

- If $\lambda \neq 0$ we continue like thisí:

$$\xrightarrow{H_{42}(-2/\lambda)} \xrightarrow{\nu_{42}(-2/\lambda)} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda + 4/\lambda \end{pmatrix}$$

The terms on the diagonal vanish in $\lambda = -2, 0, 2$. We distinguish the following cases: - If $\lambda < -2$ the signature is $(+,+,+;+)$. It is an imaginary ellipsoid. - If $\lambda = -2$ the signature is $(+,+,+;0)$. It is an imaginary cone. - If $-2 < \lambda < 0$ the signature is $(+,+,+;-)$. It is a real ellipsoid. - If $\lambda = 0$, the matrix is:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

We cannot further diagonalize. The signature of the matrix T of quadratic terms is $(+,+,0)$ and it is therefore an elliptic paraboloid. - If $0 < \lambda < 2$ the signature is $(+,+,-;+)$. It is a hyperboloid with two sheets. - If $\lambda = 2$ the signature is $(+,+,-;0)$. This is a real cone. - If $\lambda > 2$ the signature is $(+,+,-;-)$. It is a one sheet hyperboloid.

10.– *Given the quadric equation:*

$$x^2 - 8z^2 + 4xy + 2xz - 8yz + 8y + 8z + 2 = 0$$

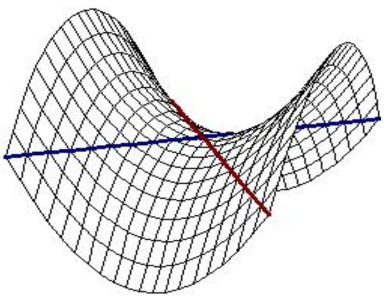
classify the surface and sketch a picture of it. The matrix associated with the quadric is:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & -4 & 4 \\ 1 & -4 & -8 & 4 \\ 0 & 4 & 4 & 2 \end{pmatrix}$$

For classify it, we diagonalize it by congruence taking into account that the ĺast row cannot be added to the others, changed by another or multiplied by a number:

$$A \xrightarrow{H_{21}(-2)} \xrightarrow{H_{31}(-1)} \xrightarrow{\mu_{21}(-2)} \xrightarrow{\mu_{31}(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -6 & 4 \\ 0 & -6 & -9 & 4 \\ 0 & 4 & 4 & 2 \end{pmatrix} \xrightarrow{H_{32}-3/2} \xrightarrow{H_{42}(1)} \xrightarrow{\mu_{32}-3/2} \xrightarrow{\mu_{42}(1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 6 \end{pmatrix}$$

Could not follow diagonalization without using last row. The type of quadric depends on the signature of the matrix of quadratic terms, which is $(+,-,0)$. It is therefore a hyperbolic paraboloid.



11.– *Given the quadric of equation:*

$$x^2 + 3y^2 + 4z^2 + 2xy + 4xz - 16yz - 12y + 12z + 3 = 0$$

classify the surface and sketch a picture of it. The matrix associated with the quaric is:

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 3 & -8 & -6 \\ 2 & -8 & 4 & 6 \\ 0 & -6 & 6 & 3 \end{pmatrix}$$

To classify it, we diagonalize it by congruence using elementary operations (the same per row as per column), with the proviso that the fourth row can neither be added to the others, neither changed position, nor multiplied by a number.

$$A \xrightarrow{H_{21}(-1)} \xrightarrow{H_{31}(-2)} \xrightarrow{\mu_{21}(-1)} \xrightarrow{\mu_{31}(-2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -10 & -6 \\ 0 & -10 & 0 & 6 \\ 0 & -6 & 6 & 3 \end{pmatrix} \xrightarrow{H_{32}(5)} \xrightarrow{H_{42}(3)} \xrightarrow{\mu_{32}(5)} \xrightarrow{\mu_{42}(3)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -50 & -24 \\ 0 & 0 & -24 & -15 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{H_{43}(-12/25)} \xrightarrow{\mu_{43}(-12/25)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -50 & 0 \\ 0 & 0 & 0 & -87/250 \end{pmatrix}$$

The signature is $(+, +, -, -)$. It is therefore a one-sheet hyperboloid.