LINEAR ALGEBRA II — Exercises Unit I, Chapter 1

Bilinear mappings and quadratic forms (Academic year 2022–2023)

1.– Find the matrix associated to each one of the following bilinear forms with respect to the canonical basis:

(a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f((x,y),(x',y')) = xx' + 3xy' - yx' + 2yy'$

(b) $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$

(c) $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad f((x,y,z),(x',y',z')) = xx' + 2xz' - 4yy' + 2zx' + 8zz'$

2.–

(a) Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$. Obtain $f((1,1),(3,2))$.

(b) Let $f : \mathcal{M}_{2\times 2}(\mathbb{R}) \times \mathcal{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$. Obtain $f\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right)$.

3.– Let C be the canonical basis of $\mathbb{R}^2$ and let $B = \{(3,2),(1,1)\}$.

(a) $F_C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is the matrix associated to a certain bilinear form relative to the basis C. Obtain its associated matrix F_B relative to the basis B.

(b) $F_B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ is the matrix associated to a certain bilinear form relative to the basis B. Obtain its associated matrix F_C relative to the basis C.

4.– Indicate whether the bilinear forms in Exercise 1 are symmetric, antisymmetric or neither one thing nor the other.

5.– Find the matrix associated to each one of the following quadratic forms relative to the canonical basis:

(a) $w : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad w(x,y) = x^2 - 4xy + 4y^2$.

(b) $w : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad w(x,y,z) = x^2 - 4xz + y^2 - 2yz - z^2$.

(c) $w : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad w(x,y,z) = x^2 + 2xy + 2xz + 2y^2 + 4yz + 10z^2$.

6.– For the quadratic form in item (a) of Exercise 5:

(a) Check if the following pairs of vectors are conjugate:

$(0,1)$ and $(1,0)$, $\quad (1,0)$ and $(2,1)$, $\quad (1,-1)$ and $(1,1)$, $\quad (0,1)$ and $(0,1)$

(b) Check if any of the following vectors is self-conjugate:

$(1,2)$, $\quad (1,1)$, $\quad (2,1)$.

7.– Find the implicit equations of the kernel of each one of the quadratic forms in Exercise 5.

8.– Find the ranks of all the quadratic forms in Exercise 5.

9.– Find the signature of, and classify, all quadratic forms in Exercise 5, indicating whether they are degenerate or not, and whether they are definite (positive or negative), semi-definite (positive or negative) or inidefinite.

Solutions.

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1. (a) $\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & 8 \end{pmatrix}$.

2. (a) 10. (b) 14.

3. (a) $F_B = \begin{pmatrix} 25 & 11 \\ 9 & 4 \end{pmatrix}$. (b) $F_C = \begin{pmatrix} 9 & -12 \\ -14 & 19 \end{pmatrix}$.

4. (a) Neither one nor the other. (b) Antisymmetric. (c) Symmetric.

5. (a) $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. (b) $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -2 & -1 & -1 \end{pmatrix}$. (c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 10 \end{pmatrix}$.

6. (a) No. Yes. No. No. (b) No. No. Yes.

7. (a) $x - 2y = 0$. (b) $x = 0,\ y = 0,\ z = 0$. (c) $x = 0,\ y = 0,\ z = 0$. .

8. (a) 1. (b) 3. (c) 3.

9.

(a) $signature = (1, 0)$, degenerate, semi-definite positive.

(b) $signature = (2, 1)$, not degenerate, indefinite.

(c) $signature = (3, 0)$, not degenerate, definite positive.