LINEAR ALGEBRA I — Problem set 3

Matrices and determinants (Academic year 2022–2023)

1.– Given the set of matrices with dimension $n \times n$ of real elements, prove that the product of lower triangular matrices is another lower triangular matrix.

(Partial Exam, February 2000)

2.– Let A be an $n \times n$ diagonal matrix where all the diagonal elements are different from each other. Prove that any $n \times n$ matrix that commutes with A should be diagonal.

(Final Exam, June 2002)

3.– Let $A, B, X \in M_{n\times n}(\mathbb{R})$ be invertible matrices, compute X in terms of A and B satisfying:

$$(A^{-1}X)^{-1} = A(B^{-2}A)^{-1}.$$

Also, prove that $sign(det(A)) = sign(det(X))$.

(Exam, October 2014)

4.– Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$ be two matrices. Determine and reason whether the following statements are true or false:

(i) $AB = 0 \Rightarrow A = 0$ or $B = 0$.

(ii) $(A + B)^2 = A^2 + 2AB + B^2$.

(iii) If C is invertible and $AB = C$ then A, B are invertibles.

(Final Exam, January 2016)

5.– Let $A \in M_{n\times n}(\mathbb{R})$ be a square matrix satisfying $A^2 + A + Id = 0$.

(i) Prove that A is invertible.

(ii) Prove that $A^{-1} = -(A + Id) = A^2$.

(iii) Compute A^3 and A^{2013}.

(Exam, October 2013)

6.– Compute the n-th power of the matrix: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

7.– Given the matrix:

$$A = \begin{pmatrix} x & x+1 & x+2 \\ x+3 & x+2 & x+1 \\ x+2 & x & x+4 \end{pmatrix}$$

(i) Find x such that $det(A) = 0$.

(ii) Analyze the rank of A depending on the possible values of x.

(Final Exam, July 2015)

8.– Given $A \in M_{n\times n}(\mathbb{R})$ such that $A^4 = A$. Determine and reason whether the following statements are true or false:

(i) $A^3 = Id$.

(ii) $A^{34} = A$.

(iii) If A is invertible, then $det(A) = 1$.

(iv) It may happen that $A^2 = A$.

(Exam, October 2017)

9.– Let $A, B \in M_{n\times m}(\mathbb{R})$ be two matrices. Determine and reason whether the following statements are true or false:

(i) If $rank(A) = n$ then $m \geq n$.

(ii) If $rank(A) = 1$ then A has all but one null rows.

(iii) If $n = m$ then $rank(A^2) = rank(A)$.

(iv) If B is invertible, $rank(AB^t) = rank(A)$.

(Final Exam, July 2021)

10.– Let $A, B, C \in M_{6\times 6}(\mathbb{R})$ be three square matrices such that $-ABA^t = CA + A$, $det(B) = 1$, A is invertible, and C is a diagonal matrix with $c_{ii} = i$. Compute $det(A)$.

(Final Exam, January 2016)

11.– Compute the following determinants:

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 4 & 5 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 4 \\ 3 & 0 & 5 & 7 \\ 0 & 2 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 12 & 123 & 1234 \\ 2 & 23 & 234 & 2341 \\ 3 & 34 & 341 & 3412 \\ 4 & 41 & 412 & 4123 \end{vmatrix}.$$

12.– Assuming that $det\begin{pmatrix} 2 & b & 3 \\ a & 0 & 1 \\ 1 & 5 & c \end{pmatrix} = 5$, compute:

(i) $det\begin{pmatrix} 2-3a & b & 0 \\ 2 & b & 3 \\ 5 & 2b+5 & c+6 \end{pmatrix}$ (ii) $det\begin{pmatrix} b & 5 & 0 \\ 4 & c+2 & 2 \\ a+2 & 2a+1 & 2a \end{pmatrix}$.

(Final Exam, July 2020)

13.– Considering the matrix $A = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}$

(i) Compute AA^t.

(ii) Compute $det(AA^t)$ and $det(A)$.

(iii) Analyze the rank of A depending on the possible values of a, b, c, d.

(Final Exam, July 2019)

14.– Given $n \in \mathbb{N}$, the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$(P_n)_{ij} = \begin{cases} 0 \text{ if } i = j+1 \\ \\ i \text{ if } i \neq j+1 \end{cases}$$

(i) Write explicitly the matrix P_4.

(ii) Find the determinant of P_4.

(iii) Find the general expression of $det(P_n)$.

(Final Exam, January 2019)

15.– Given $n \in \mathbb{N}$, the matrix $A_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$a_{ij} = i - 2j, \qquad i, j = 1, 2, \ldots, n.$$

(i) Write explicitly the matrix A_4.

(ii) Compute $det(A_4)$.

(iii) For $n \geq 2$, compute $trace(A_n)$, $det(A_n)$ and $rank(A_n)$.

(Final Exam, January 2018)

16.– Given $n \in \mathbb{N}$, the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$(P_n)_{ij} = \begin{cases} i \text{ if } j \leq n+1-i \\ \\ 0 \text{ if } j > n+1-i \end{cases}$$

(i) Write explicitly the matrix P_5 and find its determinant.

(ii) For each $n \geq 2$, find $det(P_n)$ and $trace(P_n)$.

(Final Exam, January 2021)

17.– Given $n \in \mathbb{N}$, the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$(P_n)_{ij} = \begin{cases} 2 \text{ if } i > j \\ \\ 1 \text{ if } i \leq j \end{cases}$$

(i) Write explicitly the matrix P_4.

(ii) Compute the determinat of P_4.

(iii) For each $n \geq 2$, find $det(P_n)$, $trace(P_n)$ and $det(P_n^{2020})$.

(Exam, January 2020)

I.– Given the set of matrices with dimension $n \times n$ of real elements, prove that if $AA^T = \Omega$, then $A = \Omega$.

(Exam, February 2000)

II.– Compute the n-th power of the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}.$$

III.– For the following families of nonsingular matrices $\mathcal{M}_{n\times n}(K)$, determine if they verify the following conditions: (a) given a family's matrix, its inverse also belongs to the family; (b) given two matrices of the family, their product also belongs to the family.

(1) the regular symmetrical matrices,

(2) the regular matrices commuting with a given matrix $A \in \mathcal{M}_{n\times n}(K)$,

(3) the orthogonal matrices.

IV.– Let A be a column matrix $n \times 1$ such that $A^t A = 1$ and $B = Id_n - 2AA^t$. Prove that:

a) B is symmetric.

b) $B^{-1} = B^t$.

(Exam, January 2008)

V.– Given $n \in \mathbb{N}$, the matrix $A_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$a_{ij} = \begin{cases} 0 \text{ if } i = j \\ i \text{ if } i \neq j \end{cases}$$

(i) Write explicitly the matrix A_4.

(ii) Calculate $det(A_4)$.

(iii) Given $n \geq 2$, find $trace(A_n)$, $det(A_n)$ and $rank(A_n)$.

(Final Exam, July 2018)

VI.– Let X be a square matrix of dimensions $n \times n$ and real elements. Let k be an even number. Prove that if $X^k = -Id$, then n is also an even number.

VII.– Given the matrix A with dimension $m \times n$ with $m, n > 1$,

$$A = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n+1 & n+2 & \dots & 2n-1 & 2n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (m-1)n+1 & (m-1)n+2 & \dots & mn-1 & mn \end{pmatrix}$$

write a_{ij} in terms of i and j. Compute its rank.

(Final Exam, September 2005)

VIII.– If $A = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}$ and $det(A) = 3$, compute $det(2C^{-1})$ where $C = \begin{pmatrix} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{pmatrix}$.

(Final Exam, January 2014)

IX .– Compute the following determinant:

$$\begin{vmatrix} 0 & x_1 & x_2 & \dots & x_n \\ x_1 & 1 & 0 & \dots & 0 \\ x_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \dots & 1 \end{vmatrix}$$

Find the real values of $x_1, x_2, \dots, x_n$ such that is null.

(Final Exam, 2011)

X.– Given $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, we consider the matrix $A \in M_{n\times n}(\mathbb{R})$:

$$A = \begin{pmatrix} a+b & a & a & \dots & a & a \\ a & a+b & a & \dots & a & a \\ a & a & a+b & \dots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \dots & a+b & a \\ a & a & a & \dots & a & a+b \end{pmatrix}$$

(i) Find $det(A)$ in terms of a, b, n.

(ii) Find $rank(A)$ in terms of a, b, n.

(Exam, October 2014)

XI.– Compute the following determinant for $n \geq 2$

$$A_n = \begin{vmatrix} x_1+y_1 & x_1+y_2 & x_1+y_3 & \cdots & x_1+y_n \\ x_2+y_1 & x_2+y_2 & x_2+y_3 & \cdots & x_2+y_n \\ x_3+y_1 & x_3+y_2 & x_3+y_3 & \cdots & x_3+y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n+y_1 & x_n+y_2 & x_n+y_3 & \cdots & x_n+y_n \end{vmatrix}$$

(Exam, February 2003)

XII.– Compute the following determinant:

$$det \begin{pmatrix} 1 & x & x^2 & 1 \\ x & x^2 & 1 & 1 \\ x^2 & 1 & 1 & x \\ 1 & 1 & x & x^2 \end{pmatrix}$$

Find the real values of x such that $det(A) = 0$.

(Final Exam, 2011)

XIII.– Given $n \in \mathbb{N}$, the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$(P_n)_{ij} = \begin{cases} i \text{ si } i \leq j \\ 1 \text{ si } i > j \end{cases}$$

(i) Write explicitly the matrix P_4 and find its determinant.

(ii) For any $n \geq 2$, find $det(P_n)$, $trace(P_n)$ and $det(P_n^{-1})$.

(Exam, January 2022)

XIV.– Given $n > 2$, $A \in M_{n\times n}(\mathbb{R})$ an invertible matrix, and $adj(A)$ its adjoint matrix. Prove that:

(a) $det(adj(A)) = det(A)^{n-1}$.

(b) $adj(adj(A)) = det(A)^{n-2} \cdot A$.

(Exam, January 2010)

XV.– Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Determine and reason whether the following statements are true or false:

(i) If $rank(A) = 1$ then $rank(AB) \leq 1$.

(ii) If $rank(A) = rank(B)$ then $rank(AB) = rank(A)$.

(iii) $rank(A) + rank(B) = rank(A + B)$.

(iv) $rank(A) + rank(B) > rank(A + B)$.

(Final Exam, January 2017)
