LINEAR ALGEBRA I — EXERCISES Part III. Chapter 4

Endomorphisms. (Academic year 2022–2023)

1.– Find the matrix F_C associated to the endomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (x + 2y, 2x + 4y)$ with respect to the canonical basis.

2.– Check that the endomorphism f from the previous exercise satisfies $f(2, -1) = 0 \cdot (2, -1)$ and $f(1, 2) = 5 \cdot (1, 2)$.

3.– Find the matrix F_B associated to f with respect to the basis $B = \{(2, -1), (1, 2)\}$. Check that it is diagonal.

4.– Find the characteristic polynomial of the endomorphism f from Problem 1. Which are its roots?.

5.– Factorize the following polynomials into $\mathbb{R}$ knowing that all of them have at least one integer root:

(i) $x^2 - 5x + 6$.

(ii) $x^3 - 6x^2 + 12x - 8$.

(iii) $x^3 + x^2 - 3x - 3$.

(iv) $x^4 + x^3 - x^2 - x$.

(v) $x^3 - x^2 + x - 1$.

6.– Given the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \end{pmatrix}$:

(i) Compute its characteristic polynomial.

(ii) Find its eigenvalues and their algebraic multiplicities.

(iii) Find the geometric multiplicities of its eigenvalues.

(iv) Compute a basis of the characteristic subspace associated to each eigenvalue.

(v) Multiply the matrix A with each one of these eigenvectors. What do you notice?

(vi) Let D be the diagonal matrix whose diagonal is formed by the eigenvalues repeated as many times as their respective multiplicities; let P be the matrix whose columns are the eigenvectors in the same order as the one chosen for the eigenvectors. Check that $AP = PD$. Note that this is equivalent to $P^{-1}AP = D$.

Soluciones.

1. $F_C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

3. $F_B = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$.

4. $P_f(\lambda) = \lambda(\lambda - 5) = \lambda^2 - 5\lambda$. Its roots are 0 and 5.

5.

(i) $x^2 - 5x + 6 = (x-2)(x-3)$.

(ii) $x^3 - 6x^2 + 12x - 8 = (x-2)^3$.

(iii) $x^3 + x^2 - 3x - 3 = (x+1)(x-\sqrt{3})(x+\sqrt{3})$.

(iv) $x^4 + x^3 - x^2 - x = x(x-1)(x+1)^2$.

(v) $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$.

6.

(i) $p_A(\lambda) = \lambda^2(\lambda-2)(\lambda-5)$.

(ii) $\lambda_1 = 0$ with algebraic m.$= 2$; $\lambda_2 = 2$ with algebraic m.$= 1$; $\lambda_3 = 5$ with algebraic m.$= 1$.

(iii) geometric m.$(0) = 2$, geometric m.$(2) =$, geometric m.$(5) = 1$.

(iv)(*) $S_0 = \mathcal{L}\{(2,-2,0,1),(1,-1,1,0)\}$, $S_2 = \mathcal{L}\{(-3,-3,1,5)\}$, $S_5 = \mathcal{L}\{(0,0,1,2)\}$.

(v) $A\begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad A\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix},$

$A\begin{pmatrix} -3 \\ -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ 2 \\ 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} -3 \\ -3 \\ 1 \\ 5 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix} = 5 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$

Note that the image of each eigenvector is the same vector multiplied by its associated eigenvalue.

(vi)(*) $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & 1 & -3 & 0 \\ -2 & -1 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 5 & 2 \end{pmatrix}$.

(*) The solution is not unique, that is, there are several different correct answers.