LINEAR ALGEBRA I — Exercises Part III. Chapter 1 and 2

Vector spaces. (Academic year 2022–2023)

1.– Let $\mathcal{P}_1(\mathbb{R}) = \{p(x) = a_0 + a_1 x | a_0, a_1 \in \mathbb{R}\}$ be the set of polynomials of degree at most 1. We define the sum of polynomials as:

$$p(x) = a_0 + a_1 x, \quad q(x) = b_0 + b_1 x \Rightarrow p(x) + q(x) := (a_0 + b_0) + (a_1 + b_1)x$$

Taking $p(x) = a_0 + a_1 x$, $q(x) = b_0 + b_1 x$, $r(x) = c_0 + c_1 x$, prove the following properties:

- associative: $p(x) + (q(x) + r(x)) = (p(x) + q(x)) + r(x)$.

- commutative: $p(x) + q(x) = q(x) + p(x)$.

- identity element: the identically zero polynomial $p_0(x) = 0$ satisfying $p(x) + p_0(x) = p(x)$.

- inverse element: given any $p(x) = a_0 + a_1 x$ the polynomial $-p(x) = -a_0 - a_1 x$ satisfies $p(x) + (-p(x)) = 0$.

2.– In $\mathcal{P}_1(\mathbb{R})$ we define the product of a polynomial by a scalar:

$$p(x) = a_0 + a_1 x, \quad \lambda \in \mathbb{R} \Rightarrow \lambda \cdot p(x) := \lambda a_0 + \lambda a_1 x.$$

Taking $p(x) = a_0 + a_1 x$, $q(x) = b_0 + b_1 x$ and $\lambda, \mu \in \mathbb{R}$, prove the following properties:

- $1 \cdot p(x) = p(x)$.

- $\mu(\lambda \cdot p(x)) = (\mu\lambda) \cdot p(x)$.

- $\lambda \cdot (p(x) + q(x)) = \lambda \cdot p(x) + \lambda \cdot q(x)$.

- $(\lambda + \mu) \cdot p(x) = \lambda \cdot p(x) + \mu \cdot p(x)$.

3.– Prove that the subset of polynomials $U = \{a_0 + a_1 x \in \mathcal{P}_1(\mathbb{R}) | a_0 + a_1 = 1\}$ is NOT a vector subspace of $\mathcal{P}_1(\mathbb{R})$, by giving a vector $p(x) \in U$ and a number λ such that $\lambda \cdot p(x) \notin U$.

4.– Prove that the subset of polynomials $U = \{a_0 + a_1 x \in \mathcal{P}_1(\mathbb{R}) | a_0 + a_1 = 0\}$ IS a vector subspace of $\mathcal{P}_1(\mathbb{R})$. To this end, given $p(x) = a_0 + a_1 x$, $q(x) = b_0 + b_1 x$ such that $p(x), q(x) \in U$, that is, such that $a_0 + a_1 = 0$ and $b_0 + b_1 = 0$, verify that $\lambda \cdot p(x) + \mu \cdot q(x) \in U$ for any numbers $\lambda, \mu \in \mathbb{R}$.

5.– Applying the definition determine whether the vectors $(1, 0, 1), (2, 1, 1), (0, 1, -1)$ are linearly independent.

6.– Write the vector $(2, 3)$ as a linear combination of the vectors $(1, 0), (0, 1)$ and $(1, 1)$. Is there a unique way to do it?

7.– Write the vector $(2, 3)$ as a linear combination of the vectors $(0, 1)$ and $(1, 1)$. Is there a unique way to do it?

8.– If $(4, 3)_B$ are the coordinates of a vector with respect to the basis $B = \{(1, 1), (2, -1)\}$, which are the components of this vector as an element of $\mathbb{R}^2$?

9.– Given the vectors $(1,0,0),(1,1,0)$ find a third vector $\vec{u}$ such that $\{(1,0,0),(1,1,0),\vec{u}\}$ is a basis of $\mathbb{R}^3$.

10.– Give the canonical basis of $\mathcal{M}_{3\times 2}(\mathbb{R})$.

11.– Given the basis $B=\{(1,1),(2,3)\}$ of $\mathbb{R}^2$, give the change-of-basis matrix M_{CB}, where C is the canonical basis.

12.– In a vector space V we have the bases $B_1=\{\vec{u}_1,\vec{u}_2\}$ and $B_2=\{\vec{v}_1,\vec{v}_2\}$.

(i) If $\vec{v}_1=\vec{u}_1+\vec{u}_2$ and $\vec{v}_2=2\vec{u}_1+3\vec{u}_2$, give the change-of-basis matrices $M_{B_1B_2}$ and $M_{B_2B_1}$.

(ii) If $\vec{w}=(-1,3)_{B_2}$ write the vector $\vec{w}$ as linear combination of $\vec{v}_1$ and $\vec{v}_2$.

(iii) Write the coordinates of $\vec{w}$ respect to the basis B_1.

13.– In $\mathbb{R}^3$ given the subspace $U=\mathcal{L}\{(1,0,1),(1,1,1)\}$, find its parametric and implicit equations with respect to the canonical basis.

14.– In $\mathbb{R}^3$, given the subspace $U=\{(x,y,z)\in\mathbb{R}^3\,|\,x+y-2z=0\}$, find its parametric and implicit equations with respect to the canonical basis.

Solutions.

3$^{(*)}$**.** $p(x)=1$, $\lambda=2$.

5. They are not linearly independent: $2\cdot(1,0,1)-1\cdot(2,1,1)+1\cdot(0,1,-1)=(0,0,0)$.

6$^{(*)}$**.** $(2,3)=1\cdot(1,0)+2\cdot(0,1)+1\cdot(1,1)$. The solution is not unique.

7. $(2,3)=1\cdot(0,1)+2\cdot(1,1)$. The solution is unique.

8. $(10,1)$.

9. $(0,0,1)$. The solution is not unique.

10. $C=\left\{\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix}\right\}$.

11. $M_{CB}=\begin{pmatrix}1&2\\1&3\end{pmatrix}$.

12. (i) $M_{B_1B_2}=\begin{pmatrix}1&2\\1&3\end{pmatrix}$ and $M_{B_2B_1}=M_{B_1B_2}^{-1}=\begin{pmatrix}3&-2\\-1&1\end{pmatrix}$. (ii) $\vec{w}=-\vec{v}_1+3\vec{v}_2$. (iii) $\vec{w}=(5,8)_{B_1}$.

13$^{(*)}$**.** Parametric: $x=a+b,\quad y=b,\quad z=a+b$. Implicit: $x-z=0$.

14$^{(*)}$**.** Parametric: $x=2a+b,\quad y=-b,\quad z=a$. Implicit: $x+y-2z=0$.
